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On a New Interpolation Process

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1. Let $f(x)$ be a 2π-periodic continuous function defined on the real line. Throughout this paper, M is assumed to be a fixed positive even integer. Also, we set $x_{kn} = 2k\pi/n$, $k = 0, 1,..., n-1$. A. Sharma and the author [5] introduced the trigonometric polynomials

$$R_n(f; x) = \sum_{k=0}^{n-1} f(x_{kn}) F_M(x - x_{kn}), \tag{1.1}$$

where

$$F_M(x) = \frac{1}{n}\left[1 + 2\sum_{j=1}^{n-1} \frac{(n-j)^M}{(n-j)^M - j^M} \cos jx\right]$$

(the case $M = 2$ was considered already by O. Kis [3]). Since, for $i, k = 0, 1,..., n-1$, we have

$$F_M(x_{in} - x_{kn}) = \delta_{ik}$$

and

$$F_M^{(M)}(x_{in} - x_{kn}) = 0,$$

it follows that

$$R_n(f, x_{in}) = f(x_{in}), \qquad R_n^{(M)}(f, x_{in}) = 0 \qquad (i = 0, 1,..., n-1).$$

We showed [5, Theorem 3, page 343] that $R_n(f; x)$ converges uniformly to $f(x)$ on the real line provided $f(x)$ satisfies the Zygmund condition

$$f(x + h) - 2f(x) + f(x - h) = o(h) \tag{1.2}$$

and that this result is best possible in the sense that the Zygmund class cannot be replaced by Lipα, $0 < \alpha < 1$.

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Consider

$$\bar{R}_n(f; x) = \sum_{k=0}^{n-1} f(x_{kn}) F_{2M}(x - x_{kn}). \tag{1.3}$$

Using the properties of $F_{2M}(x)$, we observe that

$$\bar{R}_n(f, x_{kn}) = f(x_{kn}), \qquad \bar{R}_n^{(2M)}(f, x_{kn}) = 0 \qquad (k = 0, 1,..., n-1),$$

and as stated above $\bar{R}_n(f; x)$ converges uniformly to $f(x)$ provided $f(x)$ satisfies (1.2).

The object of this paper is to consider a simple combination of (1.1) and (1.3) for which one can extend the class of functions for which uniform convergence holds beyond the Zygmund class and which continues to have the interpolation property. Let

$$A_n(f; x) = 2\bar{R}_n(f, x) - R_n(f, x). \tag{1.4}$$

Then $A_n(f, x)$ is a trigonometric polynomial of order $n - 1$ at most having the interpolation property

$$A_n(f, x_{kn}) = f(x_{kn}), \qquad (k = 0, 1,..., n-1).$$

Our main result is

THEOREM. *If f is a continuous 2π-periodic function and w_f is its modulus of continuity, then*

$$|A_n(f; x) - f(x)| \leqslant C w_f\left(\frac{1}{\sqrt{n}}\right)$$

where C does not depend on x, f or n.

Remark. It is of some interest to mention here that Butzer [2] has shown that certain linear combinations of Bernstein polynomials approximate functions, under certain conditions, more closely than the Bernstein polynomials do so. Further, our operator A_n is not a positive one, so that the Bohman–Korovkin theorem is not applicable.

2. PRELIMINARIES

Let

$$H_n(x) = 2F_{2M}(x) - F_M(x) = \frac{1}{n}\left[1 + 2\sum_{j=1}^{n-1} \frac{(n-j)^M}{(n-j)^M + j^M} \cos jx\right], \tag{2.1}$$

so that

$$A_n(f; x) = \sum_{k=0}^{n-1} f(x_{kn})\, H_n(x - x_{kn}).$$

LEMMA. *There exist constants c_1 and c_2 such that*

$$\sum_{k=0}^{n-1} | H_n(x - x_{kn})| \leqslant c_1 ,$$

$$\sum_{k=0}^{n-1} \sin^2 \frac{1}{2} (x - x_{kn}) \mid H_n(x - x_{kn})| \leqslant \frac{c_2}{n} .$$

Proof. We set

$$t_n(x) = 1 + 2 \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) \cos ix, \tag{2.2}$$

which is the well-known Fejér kernel. It is known that

$$t_n(x) = \frac{1}{n} \left(\frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \right)^2 \tag{2.3}$$

and

$$\sum_{k=0}^{n-1} t_n(x - x_{kn}) = n. \tag{2.4}$$

From (2.2) follows that

$$(j + 1)\, t_{j+1}(x) - 2jt_j(x) + (j - 1)\, t_{j-1}(x) = 2 \cos jx.$$

Using this formula, we find that

$$H_n(x) = \frac{1}{n} \sum_{j=1}^{n-1} b_{nj} jt_j(x) + \frac{t_n(x)}{(n-1)^M + 1}, \tag{2.5}$$

where

$$b_{nj} = P_n(j - 1) - 2P_n(j) + P_n(j + 1)$$

and

$$P_n(y) = \frac{(n - y)^M}{(n - y)^M + y^M} .$$

Next, from (2.3) and (2.5) follows that

$$| H_n(x)| \leqslant \frac{1}{n} \sum_{j=1}^{n-1} | b_{nj} | jt_j(x) + \frac{t_n(x)}{(n-1)^M + 1}$$

and

$$\sin^2 \frac{x}{2} | H_n(x)| \leqslant \frac{1}{n} \sum_{j=1}^{n-1} | b_{nj} | + \frac{1}{n[(n-1)^M + 1]} .$$

Replacing x by $x - x_{kn}$ and summing with respect to k we find by (2.4) that

$$\sum_{k=0}^{n-1} | H_n(x - x_{kn})| \leqslant \sum_{j=1}^{n-1} | b_{nj} | j + \frac{n}{(n-1)^M + 1} \tag{2.6}$$

and

$$\sum_{k=0}^{n-1} \sin^2 \frac{1}{2} (x - x_{kn}) | H_n(x - x_{kn})| \leqslant \sum_{j=1}^{n-1} | b_{nj} | + \frac{1}{(n-1)^M + 1} . \tag{2.7}$$

We shall now estimate the b_{nj}. For this purpose we use the mean value theorem, from which it follows that

$$| b_{nj} | = | \xi_1 - \xi_2 | | P_n''(\xi)|,$$

where $j - 1 < \xi < j + 1, j - 1 < \xi_1 < j < \xi_2 < j + 1$. Also

$$\frac{1}{Mn} ((n-y)^M + y^M)^3 P_n''(y)$$
$$= (M-1) y^{M-2}(n-y)^{M-2}(2y-n)[(n-y)^M + y^M]$$
$$- 2My^{M-1}(n-y)^{M-1}[-(n-y)^{M-1} + y^{M-1}].$$

For $0 \leqslant y \leqslant n$ we have the following inequalities:

$$y(n-y) \leqslant \frac{n^2}{4}, \quad | 2y - n | \leqslant n \quad \text{and} \quad n^M \geqslant (n-y)^M + y^M \geqslant \frac{n^M}{2^{M-1}} .$$

Hence for such y we find that

$$| P_n''(y)| \leqslant \frac{M^2 2^{M+1}}{n^2} .$$

Since $|\xi_1 - \xi_2| < 2$, we have

$$|b_{nj}| < \frac{M^2 2^{M+2}}{n^2}, \qquad j = 1, 2, \ldots, n-1.$$

Substituting this into (2.6) and (2.7), we find that

$$\sum_{k=0}^{n-1} |H_n(x - x_{kn})| < M^2 2^{M+1}$$

and

$$\sum_{k=0}^{n-1} \sin^2 \frac{1}{2}(x - x_{kn}) \, |H_n(x - x_{kn})| < \frac{M^2 2^{M+2}}{n} \cdot$$

the lemma is proved.

3. Proof of the Theorem

Since

$$1 = \sum_{k=0}^{n-1} H_n(x - x_{kn}),$$

we have

$$A_n(f; x) - f(x) = \sum_{k=0}^{n-1} (f(x_{kn}) - f(x)) \, H_n(x - x_{kn}).$$

Now, it is known (see [4]) that for any $\delta > 0$ and all x and t, we have

$$|f(t) - f(x)| \leqslant \left(1 + \frac{\pi^2}{\delta^2} \sin^2 \frac{1}{2}(t - x)\right) w_f(\delta).$$

Using this inequality and the Lemma, we find that

$$|A_n(f; x) - f(x)| \leqslant w_f(\delta) \sum_{k=0}^{n-1} \left(1 + \frac{\pi^2}{\delta^2} \sin^2 \frac{1}{2}(x - x_{kn})\right) |H_n(x - x_{kn})|$$

$$\leqslant w_f(\delta) \left(c_1 + \frac{\pi^2 c_2}{\delta^2 n}\right)$$

and the Theorem is proved by choosing $\delta = 1/\sqrt{n}$.

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